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Beta Function Constraints From Renormalization Group Flows in Spin Systems

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Abstract. Inspired by previous work on the constraints that duality imposes on beta functions of spin models, we propose a consistency condition between those functions and RG flows at different points in coupling constant space. We show that this consistency holds for a non self-dual model which admits an exact RG flow, but that it is violated when the RG flow is only approximate. We discuss the use of this deviation as a test for the "goodness" of proposed RG flows in complicated models, and the use of the proposed consistency in suggesting RG equations.

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1 Introduction

Recent years have seen an explosion of work in dualities of quantum field theories, and it was to be expected that such work would have implications for Statistical Mechanics, especially when spin systems are concerned, since they are very simple avatars of QFTs.

This has indeed happened, and the work reported below has its basis in the proposals of Daamgard and Haagensen [1]. They established a relation between the beta function and (exact) dualities of spin models, then proceeded to apply this relation to obtain a transparent connection between self-dual points and phase transitions.

Here we report on perhaps the most immediate extension of that work: instead of considering dualities, we will look at spin systems having an exact RG flow. A consistency relation for the beta functions (similar to that of dualities) is then derived, and verified for one such model. We then address the case of models without exact RG flows, first by considering an artificial perturbation of an exact RG, and then by examining an actual case found in the literature. Not surprisingly, the consistency conditions are no longer obeyed. We also mention the possibility of using beta function consistency to suggest new RG schemes, and provide a simple example. We conclude by discussing possible applications of the results derived.

2 Consistency Conditions

Consider a spin system with coupling constants $\mathbf{K} = (K_1, K_2, \ldots, K_N)$, and an exact RG flow which takes the system from point $\mathbf{K}$ to point $\mathbf{K}'$ in coupling constant space in such a way that the Hamiltonian retains its form. We write

$$K_i' = t_i(K_1, \ldots, K_N) \tag{1}$$

Defining the beta functions as describing the change of couplings associated with a change of lattice spacing

$$\beta(K_i) = -a\frac{\partial K_i}{\partial a} \tag{2}$$

and applying the operator $\partial/\partial a$ to (1) yields

$$\begin{aligned} -a\frac{\partial K_i'}{\partial a} &= -a\frac{\partial}{\partial a}t_i(K_1, \ldots, K_N) \\ &= -a\frac{\partial K_j}{\partial a}\frac{\partial t_i}{\partial k_j} \end{aligned} \tag{3}$$

where a sum over repeated indices is implied. By identifying the LHS of (3) with $\beta(k_i')$ and using (2) on the RHS, we arrive at

$$\beta(K_i') = \beta(K_j)\frac{\partial t_i}{\partial K_j} \tag{4}$$

The system of equations (4) represents a strong constraint on the possible forms of the beta functions: below we will see explicitly that it does not hold for inexact RG transformations. Notice also that the key assumption made was that the RG transformation mapped the Hamiltonian $\mathcal{H}(\mathbf{K})$ into $\mathcal{H}(\mathbf{K}')$, which is of the same form as the original one. The duality transformations are a special case, which supplement equation (1) with the condition $t(t(\mathbf{K})) = \mathbf{K}$; systems which possess both dualities and quasi-exact RG flows are a fertile ground for work on consistency constraints [2].

Before proceeding, it is worth relating the beta functions to the correlation lengths, which are more readily obtained from real-world or numerical data. Consider a spin system with coupling tuned such that $\mathbf{K} = (K_1, K_2^*, \ldots, K_N^*)$ where the K_i^* are fixed points of the RG transformation which changes lattice spacing from a to $a + \delta a$. Under such a transformation, the K_i^* are left unchanged, but $K_1 \to K_1'$, and the correlation length transforms as

$$\xi(K_1', K_2^*, \ldots, K_N*) = \xi(K_1, K_2^*, \ldots, K_N^*)\left(1 + \frac{\delta a}{a}\right) \tag{5}$$

By expanding the RHS of the previous equation in a Taylor series around K_1' and using (2) we find that

$$\beta(K_1) = -\frac{\xi(K_1, K_2^*, \ldots, K_N^*)}{\frac{\partial \xi}{\partial K_1}(K_1, K_2^*, \ldots, K_N^*)} \tag{6}$$

Hence, given an RG scheme, the beta functions can be computed if the correlation lengths are known. Another relation between beta functions can be obtained by considering

$$\xi(K_1', K_2', \ldots, K_N') = \xi(K_1, K_2, \ldots, K_N)\left(1 + \frac{\delta a}{a}\right) \tag{7}$$

and performing an N-variable Taylor expansion keeping only first order terms, leading to

$$\beta(K_i)\frac{\partial \xi}{\partial K_i}(K_1, K_2, \ldots, K_N) = -\xi(K_1, K_2, \ldots, K_N) \tag{8}$$

The above relations become cumbersome analytically when the number of coupling constants is large, but they can easily be verified for a few one-coupling systems, to which we now turn.

3 Examples

We now consider a few specific examples in order to illustrate the points made above. Perhaps the simplest example of a spin system which admits a closed-form, exact RG transformation is the one-dimensional Ising model with background and magnetic field terms, the reduced Hamiltonian of which reads

$$\mathcal{H} = -K \sum_{(ij)} s_i s_j - h \sum_i s_i - \sum_i C \tag{9}$$

By a process of decimation, exact RG equations can be extracted [3]. In order to be brief, we consider the case $C = h = 0$, which admits a one-parameter exact RG transformation defined by

$$K' = \frac{1}{2} \log(\cosh(2K)) \tag{10}$$

The fixed point of (10) is $K^* = 0$. The correlation length is

$$\xi(K) = \frac{1}{\log(\coth(K))} \tag{11}$$

and using (6) the beta function is easily seen to be

$$\beta(K) = \frac{\log(\coth(K)) \coth(K)}{1 - \coth^2(K)} \tag{12}$$

Then, considering the one-coupling version of (4) it follows that

$$\frac{\log(\coth(K')) \coth(K')}{1 - \coth^2(K')} = \frac{\log(\coth(K)) \coth(K) \tanh(2K)}{1 - \coth^2(K)} \tag{13}$$

which is easily verified to hold by substituting K' by its expression (10). Similar but messier considerations would hold for the three coupling system (9).

It is important to stress that the verification (13) was not a foregone conclusion before performing the calculation. To illustrate this, consider the artificially modified RG flow

$$K' = \frac{1}{2} \log(\cosh(2K)) + \varepsilon K \tag{14}$$

where ε is such that the iterates remain in the attraction basin of the fixed point $K^* = 0$. When applied to (4) an extra factor appears and defining the deviation from consistency as

$$\Delta = 1 - \frac{\beta(K)}{\beta(K')} \frac{\partial K'}{\partial K} \tag{15}$$

we find

$$\Delta = -\varepsilon \frac{\beta(K)}{\beta(K')} \tag{16}$$

This raises the obvious question of what happens when one considers systems which do not have an exact RG transformation. The prototypical example is the 2-D Ising model with no applied magnetic field. Decimation procedures have been derived which reproduce the observed behaviour qualitatively [4] and to good numerical accuracy [5]. However, all these schemes are approximate, since the Hamiltonians generated at each iteration step include next–to–next, next–to–next–to–next... neighbour interactions; to keep the equations manageable some truncation has to be performed. While this truncation can be performed in such a way that the fundamental requirement of Hamiltonian form invariance is respected, the RG flows are no longer exact.

For simplicity, we consider a very crude decimation procedure on the 2D Ising, the Migdal-Kadanoff [6] truncation. Its RG equation reads

$$K' = -\frac{1}{2}\log\frac{2e^{-4K}}{1+e^{-8K}} \tag{17}$$

Together with the correlation length [7]

$$\xi(K) = \frac{1}{|2K - \log\tanh K|} \tag{18}$$

we can check (4). It turns out that the deviation Δ defined by (15) evaluates to

$$\Delta = 1 - 2\tanh(K)\left(\frac{2K+\log(\tanh K)}{2+\frac{2}{\sinh(2K)}}\right)\left(\frac{2K'(K)+\log(\tanh K'(K))}{2+\frac{2}{\sinh(2K'(K))}}\right)^{-1} \tag{19}$$

which, when (17) is substituted for $K'(K)$, is seen to be non-zero. This, while far from unexpected, seems to limit the applicability of the above considerations, since there are few systems which admit an exact RG transformation. However, as was seen from the artificially modified RG flow (14), Δ can be viewed as a measure of how much a proposed RG flow differs from an exact one, and this suggests that beta function consistency could be used as an extra constraint in numerical studies of RG schemes.

One further point that can be made is that the system (4) can be used to obtain the t_i given the $\beta(K_i)$ (or, equivalently, the correlation lengths). For instance, considering the one-dimensional Ising beta function (12), we have the differential equation

$$\frac{\partial K'(K)}{\partial K} = \frac{\log(\coth(K'(K)))\coth(K'(K))}{1-\coth^2(K'(K))}\left(\frac{\log(\coth(K))\coth(K)}{1-\coth^2(K)}\right)^{-1} \tag{20}$$

the solutions of which can be written in parametric form as

$$\log\left(\frac{e^{2K}+1}{e^{2K}-1}\right)\log\left(\left(\frac{e^{2K'(K)}+1}{e^{2K'(K)}-1}\right)\right)^{-1} = C \tag{21}$$

where C is a constant, related to the behaviour of the beta function for a given point in coupling space. We see that (4), while generating a series of solutions which include the correct RG equation, does not uniquely fix the solution. It does, however, provide a strong hint that such a solution should exist, as well as suggesting its general form.

4 Conclusion, Discussion & Outlook

In conclusion, we have proposed a consistency relation between RG flows and beta functions of spin systems which is obeyed by systems having an exact RG transformation which leaves the form of the Hamiltonian unchanged. We argued heuristically that a deviation from such a consistency signals an inexact RG flow which, while quite possibly accounting for the qualitative structure of the model, will not produce a quantitative agreement with exact (or sufficient accuracy numerical) results; the deviation from consistency could provide one test of the "goodness" of proposed schemes. We also proposed that the consistency relation, coupled with information about the correlation lengths of a system, could be used to suggest general forms of RG schemes. The limitations of this method stem from the fact that some boundary conditions on the consistency relations are left unspecified, but some sort of self-consistent iterative procedure could perhaps be envisaged.

While written in the language of spin systems, it is felt that the above considerations are of interest for QFTs as well, much in the same way that the original arguments of Daamgard and Haagensen [1] have been. I hope to return to this matters in the future.

References

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